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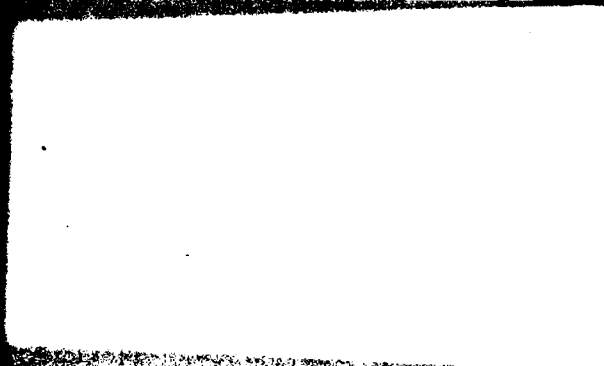
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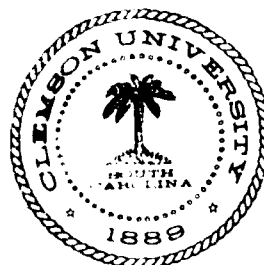
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ANALYSIS OF THE SINGLE-SERVER
QUEUE WITH UNIFORMLY LIMITED
ACTUAL WAITING TIMES BY THE
USE OF REGENERATIVE PROCESSES
AND ANALYTICAL METHODS

By

Do Le Minh*

CLEMSON UNIVERSITY

REPORT NO. N106

TECHNICAL REPORT #310

MAY, 1979

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P.O. Box 123, Broadway, New South Wales 2007, Australia.

Research supported in part by The Office of Naval Research
Contract N00014-75-C-0451

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ANALYSIS OF THE SINGLE-SERVER QUEUE WITH UNIFORMLY LIMITED
ACTUAL WAITING TIMES BY THE USE OF REGENERATIVE PROCESSES
AND ANALYTICAL METHODS.

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ABSTRACT:

This paper studies the single-server queueing system in which no customer has to wait for a duration longer than a constant K . Using analytical method together with the property that the queueing process 'starts anew' probabilistically whenever an arriving customer initiates a busy period, we obtain various transient and stationary solutions for the system.

KEY WORDS: Queueing theory. Single-server. Limited actual waiting times. Regenerative processes. Analytical methods. Actual waiting times. Lost service times. Virtual waiting times. Busy cycle. Transient results. Stationary results.

SHORT TITLE: QUEUE WITH LIMITED ACTUAL WAITING TIMES.

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1. Introduction.

In the single-server queueing model studied here, no customer has to wait for a duration longer than a constant K . If this time exceeds K , the service time of the previous customer will be so much shortened as to make it equal to K .

Cohen [8] has obtained several results for this model in which the inter-arrival times and service times have rational Laplace-Stieltjes transforms. In this paper, we shall make no specific distributional assumptions for the random variables underlying the model. Stationary results for a slightly different model in which the customers leave impatiently have been obtained by Daley [11] (See also [1], [2], [3], [4], [5], [6], [13], [14], [16], [20]).

The key to our analysis of the system is that many of its processes are regenerative; that is, they restart probabilistically whenever a customer initiates a busy period. Regenerative processes in this sense were introduced by Smith [22,23] and have been used by many authors to study the stationary behaviour of many queueing systems (See, for example, [7], [9], [10], [17], [21], [24]). By using regenerative processes along with analytical methods in this paper, we shall show that not only the stationary behaviour of the system can be studied but its transient characteristics can also be obtained. These methods also give us insight into the probabilistic structure of the system (See also [18].) In Section 3, the mathematical description of the transient behaviour of the system

is obtained from its behaviour within a busy cycle. In Section 4, the mathematical description of its stationary behaviour is also obtained from its behaviour within a busy cycle. The behaviour of the system within a busy cycle and the stochastic laws for the busy cycle are then studied in Section 5. In Section 6, due to the special structure of the M/G/1 queue, we shall obtain explicit results for the stochastic laws for the busy cycle in this system.

2. The formal model and notation.

We are given

- (D.1) a real, positive number K ;
- (D.2) an integer-valued, non-negative random variable m_0 , $E\{m_0\} < \infty$;
- (D.3) a real, non-negative random variable w_0 , $P\{w_0 \leq K\} = 1$;
- (D.4) Two independent sequences of independent and identically distributed, real, positive random variables $\{t_k, k \geq 1\}$ and $\{s_k, k \geq 1\}$. We assume that each of s_1 and t_1 has a finite first moment.

m_0 is the number of customers in the system at time $t=0-$ and w_0 is the virtual waiting time at time $t=0-$. Let customers $(m_0+1), (m_0+2), \dots, k, \dots$ arrive at the time epochs $i_{m_0+1}, i_{m_0+2}, \dots, i_k, \dots$ where $0 = i_{m_0+1} < i_{m_0+2} < \dots < i_k < \dots$. Let $i_{k+1} - i_k = t_k$, for all $k \geq m_0$.

Let the assigned service time of the k^{th} customer be s_k . This k^{th} customer will obtain full service if the $(k+1)^{\text{th}}$ customer arrives at the moment at which the work still to be handled by the server is less than K ; if it exceeds K , then we have to cut short the service time of the k^{th} customer to make the waiting time of the $(k+1)^{\text{th}}$ customer equal to k . The decision to shorten the service time of the k^{th} customer is taken at the moment of arrival of the $(k+1)^{\text{th}}$ customer.

We write

$$(D.5) \quad \phi(z) = E\{\exp(-zs_1)\} \text{ for } \operatorname{Re}(z) \geq 0;$$

$$(D.6) \quad \psi(z) = E\{\exp(-zt_1)\} \text{ for } \operatorname{Re}(z) \geq 0.$$

The customers are served in order of their arrivals and there is no limit on the size of the waiting room.

Besides \underline{I}_k , we want to study the following random variables:

(D.7) \underline{w}_k = the actual waiting time of the k^{th} customer ($k > \underline{m}_0$),

$$\begin{aligned} \text{i.e. } \underline{w}_k &= \min([\underline{w}_{k-1} + \underline{s}_{k-1} - \underline{t}_{k-1}]^+, K) \\ &= [\min(\underline{w}_{k-1} + \underline{s}_{k-1}, K + \underline{t}_{k-1}) - \underline{t}_{k-1}]^+ \end{aligned}$$

$$\text{where } [x]^+ = \max(x, 0);$$

(D.8) \underline{w}_∞ = limit in distribution of \underline{w}_k when $k \rightarrow \infty$, if this exists;

(D.9) \underline{z}_k = the lost service time of the k^{th} customer ($k > \underline{m}_0$),

$$\begin{aligned} \text{i.e. } \underline{z}_k &= \max(\underline{w}_k + \underline{s}_k - \underline{t}_k, K) - K \\ &= \max(\underline{w}_k + \underline{s}_k, K + \underline{t}_k) - K - \underline{t}_k; \end{aligned}$$

(D.10) \underline{z}_∞ = limit in distribution of \underline{z}_k when $k \rightarrow \infty$, if this exists;

(D.11) \underline{p}_1 = the duration of the initial busy period;

(D.12) \underline{p}_v = the duration of the v^{th} busy period, $v \geq 2$;

(D.13) \underline{i}_1 = the duration of the first idle period;

(D.14) \underline{i}_v = the duration of the v^{th} idle period, $v \geq 2$;

(D.15) $\underline{c}_1 = \underline{p}_1 + \underline{i}_1$ = the duration of the initial busy cycle;

(D.16) $\underline{c}_v = \underline{p}_v + \underline{i}_v$ = the duration of the v^{th} busy cycle, $v \geq 2$;

(D.17) \underline{n}_1 = the number of customers served during the initial busy period, including the \underline{m}_0 customers in the system at time $t=0^-$;

(D.18) \underline{n}_v = the number of customers served during the v^{th} busy period, $v \geq 2$;

(D.19) $\underline{y}(t)$ = the virtual waiting time at time t , $t \geq 0$,

$$\text{i.e. } \underline{y}(t) = \underline{I}_k + \min(\underline{w}_k + \underline{s}_k, K + \underline{t}_k) - t \quad \text{for } \underline{I}_k < t < \underline{I}_{k+1};$$

(D.20) $\underline{y}(\infty)$ = limit in distribution of $\underline{y}(t)$ when $t \rightarrow \infty$, if this exists;

(D.21) $\underline{N}(t)$ = the total number of customers arriving during the time interval $[0, t]$, including the \underline{m}_0 customers in the system at time $t=0^-$;

(D.22) $\underline{a}(t)$ = the time difference between t and the time of the first arrival during the interval (t, ∞) .

The results will be expressed in the following forms:

$$(D.23) \quad W_{m,w}(x, \xi, z) = \sum_{k=m+1}^{\infty} x^k E\{\exp(-\xi \underline{c}_k - z \underline{w}_k) \mid \underline{m}_0 = m, \underline{w}_0 = w\}$$

for $0 < |x| < 1, \operatorname{Re}(\xi) \geq 0, |z| < \infty, m \geq 0, w \geq 0$;

$$(D.24) \quad L_{m,w}(x, \xi, z) = \sum_{k=m+1}^{\infty} x^k E\{\exp(-\xi \underline{c}_k - z \underline{l}_k) \mid \underline{m}_0 = m, \underline{w}_0 = w\}$$

for $0 < |x| < 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \geq 0, m \geq 0, w \geq 0$;

$$(D.25) \quad V_{m,w}(x, \xi, z, s) = \int_0^{\infty} \exp(-\xi t) E\{x^{\underline{c}(t)} \exp(-z \underline{v}(t) - s \underline{a}(t)) \mid \underline{m}_0 = m, \underline{w}_0 = w\} dt$$

for $0 < |x| < 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \geq 0, \operatorname{Re}(s) \geq 0, m \geq 0, w \geq 0$;

$$(D.26) \quad C_{m,w}(x, \xi, -z) = E\{x^{\underline{n}_1} \exp(-\xi \underline{c}_1 + z \underline{l}_1) \mid \underline{m}_0 = m, \underline{w}_0 = w\}$$

for $0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \leq 0, m \geq 0, w \geq 0$;

$$(D.27) \quad W_{\infty}(z) = E\{\exp(-z \underline{w}_{\infty})\} \quad \text{for } |z| < \infty;$$

$$(D.28) \quad L_{\infty}(z) = E\{\exp(-z \underline{l}_{\infty})\} \quad \text{for } \operatorname{Re}(z) \geq 0;$$

$$(D.29) \quad V_{\infty}(z) = E\{\exp(-z \underline{v}(\infty))\} \quad \text{for } \operatorname{Re}(z) > 0.$$

We shall need the following intermediate Laplace-Stieltjes transforms:

$$(D.30) \quad \hat{W}_{m,w}(x, \xi, z) = E\left\{\sum_{k=m+1}^{\underline{n}_1} x^k \exp(-\xi \underline{c}_k - z \underline{w}_k) \mid \underline{m}_0 = m, \underline{w}_0 = w\right\}$$

for $0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, |z| < \infty, m \geq 0, w \geq 0$;

$$(D.31) \quad \hat{L}_{m,w}(x, \xi, z) = E\left\{\sum_{k=m+1}^{\underline{n}_1} x^k \exp(-\xi \underline{c}_k - z \underline{l}_k) \mid \underline{m}_0 = m, \underline{w}_0 = w\right\}$$

for $0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \geq 0, m \geq 0, w \geq 0$;

$$(D.32) \quad \hat{V}_{m,w}(x, \xi, z, s) = E\left\{\sum_{k=0}^{\underline{c}_1} x^k \exp(-\xi t - z \underline{v}(t) - s \underline{a}(t)) \mid \underline{m}_0 = m, \underline{w}_0 = w\right\}$$

for $0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \geq 0, \operatorname{Re}(s) \geq 0, m \geq 0, w \geq 0$;

$$(D.33) \quad P_{m,w}(x, \xi, s) = E \left\{ x^{\underline{n}_1} \exp(-\xi \underline{p}_1 - s \underline{i}_1) \mid \underline{m}_0 = m, \underline{w}_0 = w \right\}$$

for $0 < x \leq 1$, $\operatorname{Re}(\xi) \geq 0$, $\operatorname{Re}(s) \geq 0$, $m \geq 0$, $w \geq 0$.

It is important to point out here that there are two types of busy periods which Cohen [8, p.284] calls strong and weak busy periods. While two consecutive strong busy periods are separated with probability one by an idle period of non-zero duration; a weak busy period may be followed by an idle period of zero duration. In other words, if the $(k+1)^{\text{th}}$ customer arrives at the instant the k^{th} customer departs, the strong busy period continues while the weak busy period terminates and a new weak busy period starts. We consider both types of busy periods in this paper. If the result is applicable for both, no notational distinctions are made. If a result is applicable to the strong busy period only, then a superscript "s" is added to the notation. If a result is applicable to the weak busy period only, then the superscript "w" is added.

Remarks:

(R.1) From (D.30) and (D.31), we have

$$(2.1) \quad E(\underline{n}_1 \mid \underline{m}_0 = m, \underline{w}_0 = w) = \hat{W}_{m,w}(1, 0, 0) + m = \hat{L}_{m,w}(1, 0, 0) - m;$$

$$(2.2) \quad E(\underline{n}_2) = \hat{W}_{0,0}(1, 0, 0) = \hat{L}_{0,0}(1, 0, 0).$$

3. Regenerative results for the transient behaviour.

In this section, we shall show that the study of the actual waiting time of each customer, the lost service time of each customer and the virtual waiting time at each epoch can be reduced to the study of the lost service times within one busy cycle alone; that is, for $0 < |x| < 1$, $\text{Re}(\xi) \geq \text{Re}(s) \geq 0$, $m \geq 0$, $w \geq 0$, $W_{m,w}(x, \xi, z)$ ($|z| < \infty$), $L_{m,w}(x, \xi, z)$ ($\text{Re}(z) \geq 0$) and $V_{m,w}(x, \xi, z, s)$ ($\text{Re}(z) \geq 0$) can be obtained from $\hat{L}_{m,w}(x, \xi, z)$ ($\text{Re}(z) \geq 0$) and $C_{m,w}(x, \xi, 0)$.

The arguments are based mainly on the regenerative property that the continuations of many processes in this system beyond the end of a busy cycle are the probabilistic replicas of these processes commencing at the beginning of that busy cycle.

THEOREM 1 For $0 < |x| < 1$, $\text{Re}(\xi) \geq 0$, $|z| < \infty$, $m > 0$, $w > 0$,

$$(3.1) \quad W_{0,0}(x, \xi, z) = \hat{W}_{0,0}(x, \xi, z) / [1 - C_{0,0}(x, \xi, 0)] ;$$

$$(3.2) \quad W_{m,w}(x, \xi, z) = \hat{W}_{m,w}(x, \xi, z) + C_{m,w}(x, \xi, 0) W_{0,0}(x, \xi, z) .$$

PROOF: We have , from (D.23) and (D.30), for $m > 0$, $w > 0$,

$$(3.3) \quad W_{m,w}(x, \xi, z) = \hat{W}_{m,w}(x, \xi, z) + E \left\{ \sum_{k=1}^{\infty} x^{\frac{n_1}{n_1} + k} \exp(-\xi \tau_{\frac{n_1}{n_1} + k} - z w_{\frac{n_1}{n_1} + k}) \mid \underline{m}_0 = m, \underline{w}_0 = w \right\} .$$

Now since the queueing process starts again probabilistically when the $(\underline{n}_1 + 1)^{\text{th}}$ customer initiates the second busy period at time $\tau_{\underline{n}_1 + 1} = \underline{c}_1$, \underline{c}_1 and \underline{n}_1 are independent of $w_{\underline{n}_1 + k}$ for all $k > 1$. Also, for all $k > 1$, $w_{\underline{n}_1 + k}$ are independent of \underline{m}_0 and \underline{w}_0 .

Thus we can write

$$(3.4) \quad W_{m,w}(x, \xi, z) = \hat{W}_{m,w}(x, \xi, z) + C_{m,w}(x, \xi, 0) E \left\{ \sum_{k=1}^{\infty} x^k \exp(-\xi \underline{t}_k - z w_{\underline{n}_1+k}) \mid \underline{m}_0=0, \underline{w}_0=0 \right\}.$$

Furthermore, if $\underline{m}_0=0$ and $\underline{w}_0=0$, then $w_{\underline{n}_1+k}$ will have the same distribution as w_k . Thus (3.2) is proved. As (3.2) is also applicable when $\underline{m}_0=0$, $\underline{w}_0=0$, (3.1) follows. \square

THEOREM 2: For $0 < |x| < 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $m > 0$, $w > 0$,

$$(3.5) \quad L_{0,0}(x, \xi, z) = \hat{L}_{0,0}(x, \xi, z) / [1 - C_{0,0}(x, \xi, 0)] ;$$

$$(3.6) \quad L_{m,w}(x, \xi, z) = \hat{L}_{m,w}(x, \xi, z) + C_{m,w}(x, \xi, 0) L_{0,0}(x, \xi, z) .$$

PROOF: The proof is omitted because it is analogous to that of Theorem 1. \square

Theorems 1 and 2 relate $W_{m,w}(x, \xi, z)$ and $L_{m,w}(x, \xi, z)$ to $\hat{W}_{m,w}(x, \xi, z)$, $\hat{L}_{m,w}(x, \xi, z)$ and $C_{m,w}(x, \xi, 0)$. We shall now show that $V_{m,w}(x, \xi, z, s)$ can be obtained from $W_{m,w}(x, \xi, z)$ and $L_{m,w}(x, \xi, z)$. This is an important relation which is of interest by itself because it enables us to find the mathematical description of the behaviour of a queue in continuous time if we know its behaviour at a certain set of discrete-time epochs.

LEMMA 1: For $0 < |x| < 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $\text{Re}(s) \geq 0$, $m > 0$, $w > 0$,

$$(3.7) \quad V_{0,0}(x, \xi, z, s) = \hat{V}_{0,0}(x, \xi, z, s) / [1 - C_{0,0}(x, \xi, 0)]$$

$$(3.8) \quad V_{m,w}(x, \xi, z, s) = \hat{V}_{m,w}(x, \xi, z, s) + C_{m,w}(x, \xi, 0) V_{0,0}(x, \xi, z, s)$$

PROOF: The proof of this lemma is omitted because it is analogous

to that of Theorem 1. Here, we use the property that if $\underline{m}_0=0$ then for all $t \geq 0$, $\underline{v}(t+\underline{c}_1)$, $\underline{v}(t)$ and $\underline{a}(t+\underline{c}_1)$ have the same distributions as $\underline{v}(t)+\underline{n}_1$, $\underline{v}(t)$ and $\underline{a}(t)$ respectively. \square

LEMMA 2: For $0 < |x| \leq 1$, $\text{Re}(\xi) \geq \text{Re}(s) \geq 0$, $\text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$,

$$\begin{aligned}
 (3.9) \quad (z+s-\xi) \hat{V}_{m,w}(x, \xi, z, s) &= -x^m \exp(-zw) - zx^m \exp((s-\xi)w)/(s-\xi) \\
 &+ \left[1-x^\psi(z) \Omega(s) \right] \hat{W}_{m,w}(x, \xi, z)/x \\
 &+ z \left[1-x^\psi(\xi-s) \Omega(s) \right] \hat{W}_{m,w}(x, \xi, \xi-s)/x(s-\xi) \\
 &+ \exp(-zK) \Omega(z+s) \left[\hat{L}_{m,w}(x, \xi, z) - \hat{L}_{m,w}(x, \xi, 0) \right] \\
 &+ z \exp((s-\xi)K) \Omega(\xi) \left[\hat{L}_{m,w}(x, \xi, \xi-s) - \hat{L}_{m,w}(x, \xi, 0) \right] / (s-\xi) \\
 &+ (z+s-\xi) C_{m,w}(x, \xi, 0) / (s-\xi)
 \end{aligned}$$

PROOF: For $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(s) \geq 0$, $\text{Re}(z) \geq 0$, $0 \leq m < (\underline{n}_1-1)$, $w \geq 0$,

$$\begin{aligned}
 (3.10) \quad \hat{V}_{m,w}(x, \xi, z, s) &= E \left\{ \sum_{k=m+1}^{\underline{n}_1-1} x^k \int_{\underline{t}_k}^{\underline{t}_{k+1}} \exp(-\xi t - z(\underline{t}_k + \min(\underline{w}_k + \underline{s}_k, K + \underline{t}_k) - t) - s(\underline{t}_{k+1} - t)) dt \mid \underline{m}_0 = m, \underline{w}_0 = w \right\} \\
 &+ E \left\{ x^{\underline{n}_1} \int_{\underline{t}_{\underline{n}_1}}^{\underline{p}_1} \exp(-\xi t - z(\underline{t}_{\underline{n}_1} + \underline{w}_{\underline{n}_1} + \underline{s}_{\underline{n}_1} - t) - s(\underline{t}_{\underline{n}_1+1} - t)) dt \mid \underline{m}_0 = m, \underline{w}_0 = w \right\} \\
 &- E \left\{ x^{\underline{n}_1} \int_{\underline{p}_1}^{\underline{c}_1} \exp(-\xi t - s(\underline{t}_{\underline{n}_1+1} - t)) dt \mid \underline{m}_0 = m, \underline{w}_0 = w \right\}
 \end{aligned}$$

As $\underline{w}_{k+1} = \underline{t}_k - \min(\underline{w}_k + \underline{s}_k, K + \underline{t}_k) - \underline{t}_{k+1}$ for $k < \underline{n}_1$ and

$$\underline{p}_1 = \underline{t}_{\underline{n}_1} - \underline{w}_{\underline{n}_1} + \underline{s}_{\underline{n}_1}, \quad (3.10) \text{ becomes}$$

$$\begin{aligned}
 (3.11) \quad \hat{V}_{m,w}(x, \xi, z, s) = & \frac{1}{z+s-\xi} E \left\{ \sum_{k=m+1}^{n_1-1} x^k \left[\exp(-\xi \tau_{k+1} - z w_{k+1}) \right. \right. \\
 & \left. \left. - \exp(-\xi \tau_k - z(\min(w_k + s_k, K + t_k)) - s t_k) \right] \middle| m_0 = m, w_0 = w \right\} \\
 & + \frac{1}{z+s-\xi} E \left\{ x^{n_1} \left[\exp(-\xi p_1 - s i_1) \right. \right. \\
 & \left. \left. - \exp(-\xi \tau_{n_1} - z(w_{n_1} + s_{n_1}) - s t_{n_1}) \right] \middle| m_0 = m, w_0 = w \right\} \\
 & + \frac{1}{s-\xi} E \left\{ x^{n_1} \left[\exp(-\xi c_1) - \exp(-\xi p_1 - s i_1) \right] \middle| m_0 = m, w_0 = w \right\}.
 \end{aligned}$$

Now observe that for $k \leq n_1$, $t_k = \max(w_k + s_k, K + t_k) - K - t_k$, and hence

$$\begin{aligned}
 (3.12) \quad \exp(-\xi \tau_k - z(\min(w_k + s_k, K + t_k)) - s t_k) & \\
 = \exp(-\xi \tau_k - z(w_k + s_k) - s t_k) + \exp(-\xi \tau_k - zK - (s+z)t_k) & \\
 - \exp(-\xi \tau_k - z(\max(w_k + s_k, K + t_k)) - s t_k) & \\
 = \exp(-zs_k) \exp(-s t_k) \exp(-\xi \tau_k - z w_k) & \\
 - \exp(-zK) \exp(-(z+s)t_k) \left[\exp(-\xi \tau_k - z t_k) - \exp(-\xi \tau_k) \right] &
 \end{aligned}$$

Thus (3.11) can now be written as

$$\begin{aligned}
 (3.13) \quad \hat{V}_{m,w}(x, \xi, z, s) = & -x^m \exp(-zw) / (z+s-\xi) \\
 & + \left[1 - x \psi(z) \Omega(s) \right] \hat{W}_{m,w}(x, \xi, z) / x(z+s-\xi) \\
 & + \exp(-zK) \Omega(z+s) \left[\hat{L}_{m,w}(x, \xi, z) - \hat{L}_{m,w}(x, \xi, 0) \right] / (z+s-\xi) \\
 & + P_{m,w}(x, \xi, s) / (z+s-\xi) \\
 & + \left[C_{m,w}(x, \xi, 0) - P_{m,w}(x, \xi, s) \right] / (s-\xi)
 \end{aligned}$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $\text{Re}(s) \geq 0$, $0 \leq m \leq (n_1 - 1)$, $w \geq 0$.

It is easy to prove that (3.13) is also applicable when $n_1 = m+1$. Now since $\hat{V}_{m,w}(x, \xi, z, s)$, $P_{m,w}(x, \xi, s)$ and $C_{m,w}(x, \xi, 0)$ are analytic for $0 < |x| \leq 1$, $\operatorname{Re}(\xi) \geq \operatorname{Re}(s) \geq 0$, $\operatorname{Re}(z) \geq 0$, letting $z = \xi - s$ in (3.13) yields

$$(3.14) \quad P_{m,w}(x, \xi, s) = x^m \exp((s-\xi)w) - \left[1 - x\psi(\xi-s)\Omega(s) \right] \hat{W}_{m,w}(x, \xi, \xi-s)/x - \exp((s-\xi)K)\Omega(\xi) \left[\hat{L}_{m,w}(x, \xi, \xi-s) - \hat{L}_{m,w}(x, \xi, 0) \right]$$

If we substitute this equation back into (3.13), we obtain (3.9).

THEOREM 3: For $0 < |x| \leq 1$, $\operatorname{Re}(\xi) \geq \operatorname{Re}(s) \geq 0$, $\operatorname{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$,

$$(3.15) \quad (z+s-\xi)V_{m,w}(x, \xi, z, s) = -x^m \exp(-zw) - zx^m \exp((s-\xi)w)/(s-\xi) + \left[1 - x\psi(z)\Omega(s) \right] W_{m,w}(x, \xi, z)/x + z \left[1 - x\psi(\xi-s)\Omega(s) \right] W_{m,w}(x, \xi, \xi-s)/x(s-\xi) + \exp(-zK)\Omega(z+s) \left[L_{m,w}(x, \xi, z) - L_{m,w}(x, \xi, 0) \right] + z \exp((s-\xi)K)\Omega(\xi) \left[L_{m,w}(x, \xi, \xi-s) - L_{m,w}(x, \xi, 0) \right]/(s-\xi)$$

PROOF: The proof is straightforward from Theorems 1, 2 and Lemmas 1, 2. \square

It remains to show that $\hat{W}_{m,w}(x, \xi, z)$ can be obtained from $L_{m,w}(x, \xi, z)$ and $C_{m,w}(x, \xi, -z)$.

THEOREM 4: For $0 < |x| \leq 1$, $\operatorname{Re}(\xi) \geq \operatorname{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$,

$$(3.16) \quad [1 - x\psi(z)\Omega(\xi-z)] \hat{W}_{m,w}(x, \xi, z) = x^{m+1} \exp(-zw) - x C_{m,w}(x, \xi, -z) - x \exp(-zK)\Omega(\xi) \left[L_{m,w}(x, \xi, z) - L_{m,w}(x, \xi, 0) \right]$$

PROOF: Since $\hat{V}_{m,w}(x, \xi, z, s)$, $C_{m,w}(x, \xi, 0)$ and $P_{m,w}(x, \xi, s)$ are analytic for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq \text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$, (3.16) is obtained by putting $s = \xi - z$ in (3.13) \square

Remarks:

(R.2) (3.1) and (3.7) are the generalizations of (3.3) and (4.6) in [9, pps. 6, 13] respectively.

(R.3) When $K \rightarrow \infty$, then Theorem 3 becomes Theorem 2 in [26]. While Takacs derived the latter directly, the former is obtained via Lemma 2, which will also be useful in the derivation of Theorem 6 later.

(R.4) If we let $x \rightarrow 1$, $z \rightarrow 0$, $s \rightarrow 0$ in (3.14) and (3.16) and then use l'Hospital's Rule to obtain the limit when $\xi \rightarrow 0$, we shall obtain the following Generalized Wald's Lemma:

$$(3.17) \quad E\{p_1\} = E\{w_0\} + [E\{n_1\} - E\{m_0\}]E\{s_1\} - E\left\{\sum_{k=m+1}^{n_1} c_k\right\};$$

$$(3.18) \quad E\{c_1\} = [E\{n_1\} - E\{m_0\}]E\{t_1\}.$$

4. Regenerative results for the stationary behaviour.

For the queueing system studied in this paper, it has been proved that each of the processes $\{\underline{w}_k, k \geq \underline{m}_0\}$, $\{\underline{z}_k, k \geq \underline{m}_0\}$ and $\{\underline{v}(t), t \geq 0\}$ has a unique stationary distribution which is independent of the initial conditions [8]. In this section, we shall show that the study of the stationary behaviour of the system can also be reduced to the study of the lost service times within one busy cycle alone; that is, the expressions for $W_\infty(z)$ ($|z| < \infty$), $L_\infty(z)$ ($\text{Re}(z) \geq 0$) and $V_\infty(z)$ ($\text{Re}(z) \geq 0$) can be obtained from $\hat{L}_{0,0}(1,0,z)$. Here, we shall use a general theorem in the literature stating that the stationary distribution of a regenerative process, if it exists, is the 'time average' or 'customer average' of the process over a regenerative cycle. (See [9])

This allows us to state the following theorem without proof:

THEOREM 5:

$$(4.1) \quad W_\infty(z) = \hat{W}_{0,0}(1,0,z)/E\{\underline{n}_2\} \quad \text{for } |z| < \infty ;$$

$$(4.2) \quad L_\infty(z) = \hat{L}_{0,0}(1,0,z)/E\{\underline{n}_2\} \quad \text{for } \text{Re}(z) \geq 0 .$$

Together with (3.16), (2.2), the assertion for $W_\infty(z)$ and $L_\infty(z)$ is now true. The next theorem will allow us to find the distribution function of the stationary virtual waiting time in terms of the distribution functions of the stationary actual waiting time and the stationary lost service time.

THEOREM 6: For $\text{Re}(z) \geq 0$,

$$(4.3) \quad V_{\infty}(z) = 1 - E\{p_2\}/E\{c_2\} + \{1 - \psi(z)\}W_{\infty}(z)/zE\{t_1\} \\ + \exp(-zK)\Omega(z)[L_{\infty}(z)-1]/zE\{t_1\}$$

PROOF: This is because $V_{\infty}(z) = \hat{V}_{0,0}(1,0,z,0)/E\{c_2\}$. Upon applying l'Hospital rule to (3.13), we prove the theorem. \square

Remarks:

(R.5) When $K \rightarrow \infty$, then (4.3) becomes a well-known result due to Takacs [25] for the classical GI/G/1 queue.

5. The stochastic laws for the busy cycles.

Let

(D.34) M_z = the set of all those functions of z which are analytic in the domain $\text{Re}(z) > 0$ and continuous, free from zeros, uniformly bounded in $\text{Re}(z) \geq 0$;

(D.35) N_z = the set of all those functions of z which are analytic in the domain $\text{Re}(z) < 0$ and continuous, free from zeros, uniformly bounded for $\text{Re}(z) \leq 0$;

(D.36) R_z = the set of all those functions $\phi(z)$ which are defined for $\text{Re}(z) = 0$ on the complex plane and can be represented in the form

$$(5.1) \quad \phi(z) = E\{\underline{\zeta} \exp(-z\underline{\eta})\},$$

where $\underline{\zeta}$ is a complex (or real) random variable with $E\{|\underline{\zeta}|\} < \infty$ and $\underline{\eta}$ is a real random variable.

Let us define the following transformations on R_z :

$$(D.37) \quad T_z\{\phi(z)\} = E\{\underline{\zeta} \exp(-z\underline{\eta}^+)\};$$

$$(D.38) \quad T_z^*\{\phi(z)\} = \phi(z) - T_z\{\phi(z)\};$$

$$(D.39) \quad U_z\{\phi(z)\} = E\{\underline{\zeta} \delta(\underline{\eta} > 0) \exp(-z\underline{\eta})\};$$

$$(D.40) \quad U_z^*\{\phi(z)\} = \phi(z) - U_z\{\phi(z)\} = E\{\underline{\zeta} \delta(\underline{\eta} < 0) \exp(-z\underline{\eta})\};$$

$$(D.41) \quad V_z\{\phi(z)\} = E\{\underline{\zeta} \delta(\underline{\eta} > 0) \exp(-z\underline{\eta})\};$$

$$(D.42) \quad V_z^*\{\phi(z)\} = \phi(z) - V_z\{\phi(z)\} = E\{\underline{\zeta} \delta(\underline{\eta} \leq 0) \exp(-z\underline{\eta})\},$$

where $\delta(A)$ is the indicator function of any event A ; that is, $\delta(A) = 1$ if A occurs and $\delta(A) = 0$ if A does not occur.

Clearly, $T_z\{\phi(z)\}$, $U_z\{\phi(z)\}$ and $V_z\{\phi(z)\}$ belong to M_z and $T_z^*\{\phi(z)\}$, $U_z^*\{\phi(z)\}$ and $V_z^*\{\phi(z)\}$ belong to N_z . Also, it is easy to show that

$$(5.2) \quad U_z\{\phi(z)\} = T_z\{\phi(z)\} + \lim_{z \rightarrow -\infty} [T_z^*\{\phi(z)\}] ;$$

$$(5.3) \quad U_z^*\{\phi(z)\} = T_z^*\{\phi(z)\} - \lim_{z \rightarrow -\infty} [T_z^*\{\phi(z)\}] ;$$

$$(5.4) \quad V_z\{\phi(z)\} = T_z\{\phi(z)\} - \lim_{z \rightarrow \infty} [T_z\{\phi(z)\}] ;$$

$$(5.5) \quad V_z^*\{\phi(z)\} = T_z^*\{\phi(z)\} + \lim_{z \rightarrow \infty} [T_z\{\phi(z)\}] ;$$

$$(5.6) \quad T_z\{\phi(z)\} = U_z\{\phi(z)\} + \lim_{z \rightarrow 0} [U_z^*\{\phi(z)\}] ;$$

$$(5.7) \quad T_z\{\phi(z)\} = V_z\{\phi(z)\} + \lim_{z \rightarrow 0} [V_z^*\{\phi(z)\}] .$$

This means that the closed form expressions for these transformations can be obtained if that for $T_z\{\phi(z)\}$ is known. The following lemma, which is due to Takács [27], will enable us to obtain $T_z\{\phi(z)\}$ explicitly:

LEMMA 3: If $\phi(z) \in R_z$, then for $\text{Re}(z) > 0$, we have

$$(5.8) \quad T_z\{\phi(z)\} = \frac{1}{2\pi i} \phi(0) + \lim_{\varepsilon \rightarrow 0} \frac{z}{2\pi i} \int_{L_\varepsilon} \frac{\phi(s)}{s(z-s)} ds ,$$

where the path of integration L_ε ($\varepsilon > 0$) consists of the imaginary axis from $z = -i\infty$ to $z = -i\varepsilon$ and again from $z = i\varepsilon$ to $z = i\infty$.

PROOF: See Theorem 2 in [27].

In this paper, we have shown in Sections 3 and 4 that both the transient and stationary behaviours of the system can be studied in term of the lost service times within one busy cycle alone.

In this section, analytic methods are given for finding the integral equations that would theoretically allow us to obtain results for the lost service times within a busy cycle $\hat{L}_{m,w}(x, \xi, z)$ and the stochastic laws of the busy cycle $C_{m,w}(x, \xi, -z)$ simultaneously. These equations will be expressed in terms of the transformations defined in (D.37)-(D.42).

Basically, this method simply involves the re-arrangement of (3.16) into identities whose left hand sides belong to M_z and right hand sides belong to N_z . By Liouville's Theorem, they are functions independent of z . The integral equations will be obvious when these functions are known.

First, for the sake of simplicity, let us write

$$(D.43) \quad \hat{Q}_{m,w}(x, \xi, z) = [\hat{L}_{m,w}(x, \xi, z) - \hat{L}_{m,w}(x, \xi, 0)] \Omega(\xi),$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$.

From (3.16), we have

$$(5.9) \quad \hat{L}_{m,w}(x, \xi, 0) = \hat{W}_{m,w}(x, \xi, 0) = [x^{m+1} - x C_{m,w}(x, \xi, 0)] / [1 - x \Omega(\xi)].$$

This means that $\hat{L}_{m,w}(x, \xi, z)$ will be known if $\hat{L}_{m,w}(x, \xi, z)$ and $C_{m,w}(x, \xi, -z)$ are known.

For $0 < |x| \leq 1$, $\text{Re}(\xi) \geq \text{Re}(z) \geq 0$, we now assert that

$[1 - x \Omega(z) \Omega(\xi - z)]$ can be factorized into the form

$$(5.10) \quad [1 - x \Omega(z) \Omega(\xi - z)] = g^-(x, \xi, z) g^+(x, \xi, z),$$

where $g^-(x, \xi, z) \in M_z$ and $g^+(x, \xi, z) \in N_z$.

Such factorization always exists as we can write

$$(5.11) \quad g^-(x, \xi, z) = \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \ln [1 - x \Omega(z) \Omega(\xi - z)]^n \right\}$$

$$(5.12) \quad g^+(x, \xi, z) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \ln [1 - x \Omega(z) \Omega(\xi - z)]^n \right\}.$$

In fact, $g^+(x, \xi, z)$ and $g^-(x, \xi, z)$ are determined up to a multiplicative function of x and ξ . For if we also have $[1-x^\psi(z)\Omega(\xi-z)] = h^+(x, \xi, z)/h^-(x, \xi, z)$ where $h^+(x, \xi, z) \in M_z$ and $h^-(x, \xi, z) \in N_z$ then by Liouville's Theorem $g^+(x, \xi, z)/h^+(x, \xi, z) = g^-(x, \xi, z)/h^-(x, \xi, z) = F(x, \xi)$. If $\psi(z)$ or $\Omega(z)$ is a rational function of z , then the more useful expressions of $g^+(x, \xi, z)$ and $g^-(x, \xi, z)$ have been obtained in [26] (equations 43, 44, 50 and 51).

(3.16) can now be re-arranged as

$$\begin{aligned} (5.13) \quad g^+(x, \xi, z) \hat{W}_{m,w}(x, \xi, z) &= x^{m+1} T_z \{ g^-(x, \xi, z) \exp(-zw) \} \\ &+ x T_z \{ g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}(x, \xi, z) \} \\ &= x^{m+1} T_z^* \{ g^-(x, \xi, z) \exp(-zw) \} \\ &- x T_z^* \{ g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}(x, \xi, z) \} \\ &- x g^-(x, \xi, z) C_{m,w}(x, \xi, -z) \end{aligned}$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq \text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$. As the left hand side of this equation belongs to M_z and its right hand side belongs to N_z , applications of Liouville's Theorem and analytic continuation yields

$$\begin{aligned} (5.14) \quad g^+(x, \xi, z) \hat{W}_{m,w}(x, \xi, z) &= x^{m+1} T_z \{ g^-(x, \xi, z) \exp(-zw) \} \\ &+ x T_z \{ g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}(x, \xi, z) \} = R(x, \xi) \end{aligned}$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$; and

$$\begin{aligned} (5.15) \quad x^{m+1} T_z^* \{ g^-(x, \xi, z) \exp(-zw) \} &- x T_z^* \{ g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}(x, \xi, z) \} \\ &- x g^-(x, \xi, z) C_{m,w}(x, \xi, -z) = R(x, \xi) \end{aligned}$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \leq 0$, $m \geq 0$, $w \geq 0$.

If we re-arrange (3.16) differently and then apply Liouville's Theorem and analytic continuation, we shall also obtain

$$(5.16) \quad \frac{x \hat{Q}_{m,w}^+(x, \xi, z)}{g^+(x, \xi, z)} - x^{m+1} T_z \left\{ \frac{\exp(z(K-w))}{g^+(x, \xi, z)} \right\} + x T_z \left\{ \frac{\exp(zK) C_{m,w}^+(x, \xi, -z)}{g^+(x, \xi, z)} \right\} = S(x, \xi)$$

for $0 < |x| \leq 1$, $\operatorname{Re}(\xi) \geq 0$, $\operatorname{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$; and

$$(5.17) \quad - \frac{\exp(zK) \hat{W}_{m,w}^-(x, \xi, z)}{g^-(x, \xi, z)} + x^{m+1} T_z^* \left\{ \frac{\exp(z(K-w))}{g^+(x, \xi, z)} \right\} - x T_z^* \left\{ \frac{\exp(zK) C_{m,w}^-(x, \xi, -z)}{g^+(x, \xi, z)} \right\} = S(x, \xi)$$

for $0 < |x| \leq 1$, $\operatorname{Re}(\xi) \geq 0$, $\operatorname{Re}(z) \leq 0$, $m \geq 0$, $w \geq 0$.

The expressions of $R(x, \xi)$ and $S(x, \xi)$, which are dependent on the type of busy cycle we are interested in, will enable us to find the expressions of $\hat{Q}_{m,w}^+(x, \xi, z)$ and $C_{m,w}^+(x, \xi, -z)$ as in the following theorems:

THEOREM 7: $C_{m,w}^S(x, \xi, -z)$ and $\hat{Q}_{m,w}^S(x, \xi, z)$ satisfy the following simultaneous integral equations:

$$(5.18) \quad g^-(x, \xi, z) C_{m,w}^S(x, \xi, -z) = x^m U_z^* \{ g^-(x, \xi, z) \exp(-zw) \} - U_z^* \{ g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}^S(x, \xi, z) \}$$

for $|x| \leq 1$, $\operatorname{Re}(\xi) \geq 0$, $\operatorname{Re}(z) \leq 0$, $m \geq 0$, $w \geq 0$;

$$(5.19) \quad \frac{\hat{Q}_{m,w}^S(x, \xi, z)}{g^+(x, \xi, z)} = \frac{-x^m + C_{m,w}^S(x, \xi, 0)}{g^+(x, \xi, 0)} + x^m T_z \left\{ \frac{\exp(z(K-w))}{g^+(x, \xi, z)} \right\} - T_z \left\{ \frac{\exp(zK) C_{m,w}^S(x, \xi, -z)}{g^+(x, \xi, z)} \right\}$$

$$(0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \geq 0, m \geq 0, w \geq 0)$$

PROOF: We have $p(\frac{1}{v}^S = 0) = 0$ for all $v \geq 1$. Hence

$\lim_{z \rightarrow -\infty} C_{m,w}^S(x, \xi, -z) = 0$. Thus, if we let $z \rightarrow -\infty$ in (5.15), we shall obtain

$$(5.20) \quad R^S(x, \xi) = x^{m+1} \lim_{z \rightarrow -\infty} [T_z^* \{g^-(x, \xi, z) \exp(-zw)\}] - x \lim_{z \rightarrow -\infty} [T_z^* \{g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}^S(x, \xi, z)\}]$$

Upon substituting this back into (5.15), on behalf of (5.3), we

obtain (5.13). Also, we have $\lim_{z \rightarrow 0} g^+(x, \xi, z) \neq 0$, and from

(D.43), $\lim_{z \rightarrow 0} \hat{Q}_{m,w}^S(x, \xi, z) = 0$. Thus, if we let $z \rightarrow 0$ in (5.16),

remembering that $\lim_{z \rightarrow 0} T_z \{\dot{\varphi}(z)\} = \lim_{z \rightarrow 0} \dot{\varphi}(z)$, we have

$$(5.21) \quad S(x, \xi) = \frac{-x^{m+1} + x C_{m,w}^S(x, \xi, 0)}{g^+(x, \xi, 0)}$$

Upon substituting this back into (5.16), we obtain (5.19).

THEOREM 8: $C_{m,w}^W(x, \xi, -z)$ and $\hat{Q}_{m,w}^W(x, \xi, z)$ satisfy the following simultaneous integral equations:

$$(5.22) \quad g^-(x, \xi, z) C_{m,w}^W(x, \xi, -z) = x^m V_z^* \{g^-(x, \xi, z) \exp(-zw)\} - V_z^* \{g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}^W(x, \xi, z)\}$$

$$(0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \leq 0, m > 0, w > 0)$$

$$(5.23) \quad g^-(x, \xi, z) C_{0,0}^w(x, \xi, -z) = g^-(x, \xi, z) - \lim_{z \rightarrow \infty} g^+(x, \xi, z) \\ - V_z^*(g^-(x, \xi, z) \exp(-zK) \hat{Q}_{0,0}^w(x, \xi, z)) \\ (0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \leq 0)$$

$$(5.24) \quad \frac{\hat{Q}_{m,w}^w(x, \xi, z)}{g^+(x, \xi, z)} = \frac{-x^m + C_{m,w}^w(x, \xi, 0)}{g^+(x, \xi, 0)} \\ + x^m T_z \left\{ \frac{\exp(z(K-w))}{g^+(x, \xi, z)} \right\} - T_z \left\{ \frac{\exp(zK) C_{m,w}^w(x, \xi, -z)}{g^+(x, \xi, z)} \right\} \\ (0 < |x| \leq 1, \operatorname{Re}(\xi) \geq 0, \operatorname{Re}(z) \geq 0, m \geq 0, w \geq 0)$$

PROOF: We have $p\{\underline{w}_k^w=0\} = 0$ for $0 < \underline{m}_0 < k \leq \underline{n}_1$. Hence

$\lim_{z \rightarrow \infty} \hat{W}_{m,w}^w(x, \xi, z) = x \delta_{m,0}$ where $\delta_{i,j}$ is the Kronecker delta.

Thus, for $m > 0, w > 0$, letting $z \rightarrow \infty$ in (5.14), we obtain

$$(5.25) \quad R^w(x, \xi) = -x^{m+1} \lim_{z \rightarrow \infty} [T_z\{g^-(x, \xi, z) \exp(-zw)\}] \\ + x \lim_{z \rightarrow \infty} [T_z\{g^-(x, \xi, z) \exp(-zK) \hat{Q}_{m,w}^w(x, \xi, z)\}]$$

Upon substituting this back into (5.15), on behalf of (5.5), we obtain (5.22). When $\underline{m}_0 = \underline{w}_0 = 0$, we first modify (5.14) to have the term $[\hat{W}_{0,0}^w(x, \xi, z) - x]$ included then derive (5.23) by the same method as that for (5.22). For (5.24), the proof is similar to that for (5.19). \square

Remarks:

(R.6) When $K \rightarrow \infty$, (5.18) becomes

$$(5.25) \quad C_{m,w}^S(x, \xi, -z) = x^m U_z^+(g^-(x, \xi, z) \exp(-zw)) / g^-(x, \xi, z)$$

which is equation (198) in [28]. (3.16) now can be written as

$$(5.26) \quad \hat{W}_{m,w}^S(x, \xi, z) = x^{m+1} U_z(g^-(x, \xi, z) \exp(-zw)) / g^+(x, \xi, z).$$

This equation, together with (3.1), (3.2), (5.6), (5.25) yields

$$(5.27) \quad W_{m,w}(x, \xi, z) = x^{m+1} T_z(g^-(x, \xi, z) \exp(-zw)) / g^+(x, \xi, z),$$

a well-known result due to Pollaczek [19], Kingman [15], and Takacs [26].

6. The strong M/G/1 queue.

In this section, we shall concentrate on the queueing system in which the arrival process is a Poisson process; that is, $p\{t_1 \leq t\} = 1 - \exp(-\lambda t)$ for $t \geq 0$. We shall obtain explicit expression for $C_{m,w}^S(x, \xi, -z)$. The argument is based on the property that in this system, the idle periods are exponential distributed and independent of the busy periods.

THEOREM 9:

(a) For $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \leq 0$, $m \geq 0$, $w \geq 0$,

$$(6.1) \quad C_{m,w}^S(x, \xi, -z) = \lambda P_{m,w}^S(x, \xi, 0) / (\lambda + \xi - z)$$

(b) For $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $m \geq 0$, $w \geq 0$,

$$(6.2) \quad P_{m,w}^S(x, \xi, 0) =$$

$$\begin{aligned} & x^{m+1} e^{-\theta w} \psi(\theta) + x^{m+1} e^{-\theta K} \theta \psi(\theta) g^+(x, \xi, \theta) \left[\frac{1}{2(\lambda + \xi - \lambda x)} - \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{L_\varepsilon} \frac{\exp(s(K-w)) ds}{[\lambda + \xi - s - \lambda x \psi(s)] s} \right] \\ & \frac{1 + \lambda x e^{-\theta K} \theta \psi(\theta) g^+(x, \xi, \theta)}{2(\lambda + \xi)(\lambda + \xi - \lambda x)} \left[\frac{1}{2\pi i} \int_{L_\varepsilon} \frac{\exp(zK) ds}{[\lambda + \xi - s][\lambda + \xi - s - \lambda x \psi(s)] s} \right] \end{aligned}$$

where

(D.44) $\theta \equiv \theta(x, \xi)$ is the root of the equation

$$(6.3) \quad \lambda + \xi - z - \lambda x \psi(z) = 0$$

in the domain $\text{Re}(z) \geq 0$ and

$$(6.4) \quad g^+(x, \xi, z) = [\lambda + \xi - z - \lambda x \psi(z)] / [\theta - z]$$

PROOF: If $P\{t_1 \leq t\} = 1 - \exp(-\lambda t)$ for $t \geq 0$, then for all $v \geq 1$, i_v is independent of both p_v and n_v and $E\{\exp(-zi_v)\} = \lambda/(\lambda+z)$ for $\text{Re}(z) \geq 0$. Thus we obtain (6.1). (3.16) can now be written as

$$(6.5) \quad \hat{W}_{m,w}^S(x, \xi, z) = \frac{x^{m+1}(\lambda + \xi - z)e^{-zw} - \lambda x P_{m,w}^S(x, \xi, 0) - x(\lambda + \xi - z)e^{-zK} \hat{Q}_{m,w}^S(x, \xi, z)}{\lambda + \xi - z - \lambda x \Psi(z)}$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$. Now, since $\hat{W}_{m,w}^S(x, \xi, z)$ is analytic in the domain $\text{Re}(z) > 0$, letting $z \rightarrow \theta$ as defined in (D.44) yields:

$$(6.6) \quad P_{m,w}^S(x, \xi, 0) = x^{m+1} e^{-\theta w} \Psi(\theta) - x e^{-\theta K} \Psi(\theta) \hat{Q}_{m,w}^S(x, \xi, \theta).$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $m \geq 0$, $w \geq 0$. Also, from (5.19), we can write

$$(6.7) \quad \frac{\hat{Q}_{m,w}^S(x, \xi, z)}{g^+(x, \xi, z)} = \frac{-x^m}{g^+(x, \xi, 0)} + x^m T_z \left\{ \frac{\exp(z(K-w))}{g^+(x, \xi, z)} \right\} + \lambda P_{m,w}^S(x, \xi, 0) \left[\frac{1}{g^+(x, \xi, 0)(\lambda + \xi)} - T_z \left\{ \frac{\exp(zK)}{g^+(x, \xi, z)(\lambda + \xi - z)} \right\} \right]$$

for $0 < |x| \leq 1$, $\text{Re}(\xi) \geq 0$, $\text{Re}(z) \geq 0$, $m \geq 0$, $w \geq 0$. As $g^+(x, \xi, z)$ takes the form of (6.4), we let $z \rightarrow \theta$ in (6.7) and then eliminate $\hat{Q}_{m,w}^S(x, \xi, \theta)$ from the resulting equation and (6.6) to prove (6.2).

Remarks:

(R.7) If we let $K \rightarrow \infty$, (6.2) and (6.6) will become (25) in [12], a well-known result for the residual busy period of the M/G/1 queue.

REFERENCES:

- [1] L.G. Afanas'eva and A.V. Martynov, On ergodic properties of queues with constraints, Theory Prob. Applications, 12(1967), 104-109.
- [2] C.J. Ancker and A.V. Gafarian, Queueing with impatient customers who leave at random, J. Industrial Eng. 13(1962), 86-87.
- [3] C.J. Ancker and A.V. Gafarian, Some queueing problems with balking and reneging, I. Operat. Res. 11(1963), 88-100.
- [4] D.Y. Barrer, Queueing with impatient customers and indifferent clerks, Operat. Res. 5(1957), 644-649.
- [5] D.Y. Barrer, Queueing with impatient customers and ordered service, Operat. Res., 5(1957), 650-656.
- [6] S.M. Brody, On an integro-differential equation for systems with τ -waiting, Dopov. Akad. Nauk Ukrain. SSR. 571-573. (Transl. in Selected Translations in Mathematical Statistics and Probability 4(1959), 13-16.
- [7] M. Brown and S.M. Ross, Asymptotic properties of cumulative processes, SIAM J. Appl. Math. 22(1972), 93-105.
- [8] J.W. Cohen, The Single-Server Queue (North-Holland, Amsterdam, 1969)
- [9] J.W. Cohen, On Regenerative Processes in Queueing Theory, Lecture Notes in Economics and Mathematical Systems No. 121 (Springer-Verlag, Berlin-New York, 1976)
- [10] M.A. Crane and D.L. Iglehart, Simulating stable stochastic systems I: General multi-server queues, J. Assoc. Comput. Mach. 21(1974), 103-113.
- [11] D.J. Daley, General customer impatience in the queue GI/G/1, J. Appl. Prob. 2(1965), 186-205.

- [12] S. Erlander, The remaining busy period for a single server queue with Poisson input. Operat. Res. 13(1965), 734-746.
- [13] P.D. Finch, Deterministic customer impatience in the queueing system GI/M/1, Biometrika 47(1960), 45-52.
- [14] P.D. Finch, Deterministic customer impatience in the queueing system GI/M/1; a correction, Biometrika 48(1961), 472-473.
- [15] J.F.C. Kingman, On the algebra of queues, J. Appl. Prob., 3(1966), 285-326.
- [16] I.N. Kovalenko, Some queueing problems with restrictions, Theory Prob. Applications, 6(1961), 204-208.
- [17] A.J. Lemoine, Limit theorems for generalized single server queues, Adv. Appl. Prob. 6(1974), 159-174.
- [18] D.L. Minh, Analysis of the exceptional queueing system by the use of regenerative processes and analytical methods, Math. Operat. Res. 24(1979)
- [19] F. Pollaczek, Fonctions caracteristiques de certaines répartitions definies au moyen de la notion d'ordre. Application à la théorie des attentes, C.R. Acad. Sci. Paris, 234(1952), 2334-2336.
- [20] F. Pollaczek, Sur une théorie unifiée des problèmes stochastiques soulevés par l'encombrement d'un faisceau parfait de lignes telephoniques, C.R. Acad. Sci. Paris, 254(1962), 3965-3967.
- [21] S.M. Ross, Applied Probability Models with Optimization Applications (Holden Day, San Fransisco, 1970)
- [22] W.L. Smith, On the distribution of queueing times, Proc. Cambridge Phil. Soc. A, 49(1953), 449-461.
- [23] W.L. Smith, Regenerative stochastic processes, Proc. Roy. Soc. A, 232(1955), 6-31.

- [24] S. Stidham, Jr., Regenerative processes in the theory of queues, with applications to the alternating-priority queue, Adv. Appl. Prob., 4(1972), 542-577.
- [25] L. Takács, The limiting distribution of the virtual waiting time and the queue size for a single server queue with recurrent input and general service times, Sankhyā A, 25(1963), 91-100.
- [26] L. Takács, A fundamental identity in the theory of queues, Ann. Inst. Statist. Math., 22(1970), 339-348.
- [27] L. Takacs, On a linear transformation in the theory of probability, Acta Sci. Math. (Szeged), 33(1972), 15-24.
- [28] L. Takacs, On fluctuation problems in the theory of queues, Adv. Appl. Prob. 8(1976), 548-583.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER N106	2. GOVT ACCESSION NO. AD-A083 528	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Analysis of the Single-Server Queue With Uniformly Limited Actual Waiting Times By The Use of Regenerative Processes And Analytical Methods		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER TR #310
7. AUTHOR(s) Do Le Minh		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0451
9. PERFORMING ORGANIZATION NAME AND ADDRESS Clemson University Dept. of Mathematical Sciences Clemson, South Carolina 29631		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 047-222 NR 042-271
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Code 436 434 Arlington, Va. 22217		12. REPORT DATE May, 1979
		13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Queueing theory; Single-server; Limited actual waiting times; Regenerative processes; Analytical methods; Actual waiting times; Lost service times; Virtual waiting times; Busy cycle; Transient results; Stationary results.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper studies the single-server queueing system in which no customer has to wait for a duration longer than a constant K. Using analytical method together with the property that the queueing process 'starts anew' probabilistically whenever an arriving customer initiates a busy period, we obtain various transient and stationary solutions for the system.		

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